Plastic Zones in Glassy Polymers around Cracks Approaching a Bimaterial Interface under Small-Scale Yielding

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Synopsis

A study of the plastic zones created around the tips of cracks in glassy polymers, under small scale yielding, can be confronted by using the two well-known pressure-modified yield criteria and the elastic solution of the problem. There is particular interest in the case of a cracked body reinforced by another more resisting body under plane-strain conditions. The shapes of plastic zones developed around the tips of the crack subjected to opening mode loading conditions are examined as the crack approaches perpendicularly the bimaterial interface. For the study of plastic zones the exact solution derived from Muskhelishvili's complex potentials was used. Moreover, the plastic zones are confronted using either the exact solution or Sneddon's asymptotic expansion.

INTRODUCTION

The plastic zones created near a crack tip have a great important in fracture mechanics. This is due to the fact that the extent of the plastic zones can effect on the crack propagation velocities.[1] Moreover, the dimensions of the plastic zones and the intensity of deformation of these zones define basically the prerupture process and determine the nature of fracture of the body.

The exact determination of the development of plastic zones near a crack-tip presents great difficulties. A lot of predictions hae been made so far[2-5] for the shape and position of the plastic zones. The majority of these predictions are based on the determination of an initial elastic–plastic boundary using the Mises or Tresca criterion and the asymptotic elastic solution of the problem around the crack tip. But, the generation of the first plastic strains leads to a redistribution of stresses and further to a change in the development of the plastic zone. Thus, a new plastic zone must be derived which would differ essentially from the first one. From the generation of this second plastic zone a new redistribution of stresses will take place, and therefore a new change will be observed in the development of the plastic zone. So, we have a new plastic zone which also causes a new redistribution of stresses. Namely, the exact determination of plastic zones will follow after a series of successive approximations, and, when the finally obtained, the plastic zone will not differ essentially from the zone of the previous step.

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In the case where the initial elastic-plastic boundary is very small, i.e., when the applied loads to both lips of the crack are sufficiently small, a redistribution of stresses will hardly change the shape of the initial elastic–plastic boundary. This statement is obvious and it is justified from finite element analysis.[3] The same is true if we have a strongly hardened material. (A limiting case is when the modulus of elasticity before yielding and after yielding is the same. In this case the elastic solution is perfectly correct.) When the initial obtained elastic–plastic boundary is small, i.e., the applied loads are small, a redistribution of stresses will not alter considerably the shape of plastic zones. In this case the exact position of the elastic–plastic boundary can be obtained by using a correction factor[3] q, which is very close to unity. This occurs because the needed redistribution of stresses is not very important. But, when the initial elastic–plastic boundaries are large (slightly hardened materials), and/or the applied loads are large, the first elastic–plastic boundary given by the elastic solution of the initial problem differs considerably from the exact elastic–plastic boundary obtained by an iterative process. Obviously, in this case the first elastic–plastic boundary cannot serve to predict the real one and, consequently, the use of a correction factor to it is useless. Therefore, the elastic solution can be used for the prediction of plastic zones when the applied loads cause only a small scale yielding.

The application of the elastic solution in most glassy polymers is not out of range because a slight redistribution of stresses appears in the plastic zone of these materials used as structural members. This can be explained because glassy polymers are in general strongly hardened materials, and the stress–strain curve may be approximated by two straight lines with small difference in slopes. Moreover, special treatment in glassy polymers, like rolling, preorientation, increase of molecular weight and of crosslink density, etc., can improve their mechanical behavior. On the other hand, the approximation of the elastic–plastic boundaries can be improved by using the correction factor[3] q.

It is interesting to note that if necking is presented in the glassy polymers, an important redistribution of stresses is necessary in order to obtain the plastic zone. In this case the solution given in this paper does not work.

Plastic zones in cracked infinite plane made from glassy polymers have been studied recently[4,5] by using asymptotic solutions. As is pointed out in this paper, there is an important difference between predictions made by using exact and approximate solutions. Thus, in our study the exact formulas are applied for the determination of plastic zones.

The plastic zones were studied for two pair of materials. The first pair concerns a crack in a glassy polymer reinforced by aluminium, and the second, a crack in a glassy polymer reinforced by another more resisting glassy polymer. That is, the examined problems correspond to the case of a cracked plate reinforced by a more resisting one. In particular, the two problems were confronted in the case where the crack approaches perpendicularly the bimaterial interface, in plane-strain conditions which predominate at a small region around the crack tip when the crack is coming near the interface. The same problem has been studied in a recent paper[6] by the same authors but by using plane stress conditions. Some comparisons between the results of the above and the present paper have been made.

The most interesting result of our study is the distortion of the plastic zones as the crack approaches the bimaterial interface.

ELASTIC SOLUTION OF THE PROBLEM

We consider a straight crack CD of length $2a$ in an isotropic elastic half-plane D_2 bonded to another isotropic elastic half-plane D_1 (Fig. 1) (the adhesion between the half-planes being perfect.) The crack is located along the Ox-axis perpendicular to the interface. In each half-plane D_i $(i=1,2)$ are given its shear modulus μ_i $(i=1,2)$ and its Poisson's ratio ν_i $(i=1,2)$.

This problem can be solved with the aid of Muskhelishvili's complex potentials,[7-9] which for each half-plane are given by

$$\Phi_1(z) = \frac{1-B}{2\pi}\int_c^d \frac{g(x)}{x-z}\,dx, \quad \text{Re } z < 0 \tag{1a}$$

$$\Phi_2(z) = \frac{1}{2\pi}\int_c^d \frac{g(x)}{x-z}\,dx + \frac{A}{2\pi}\int_c^d \frac{g(x)}{x+z}\,dx - \frac{A}{\pi}\int_c^d \frac{x}{(x+z)^2}\,\bar{g}(x)\,dx, \quad \text{Re } z > 0 \tag{1b}$$

$$\Psi_1(z) = \frac{A-1}{2\pi}\left[\int_c^d \frac{\bar{g}(x)}{x-z}\,dx - \int_c^d \frac{x}{(x-z)^2}g(x)\,dx\right] = \frac{A-B}{2\pi}\int_c^d \frac{x}{(x-z)^2}\,g(x)\,dx, \quad \text{Re } z < 0 \tag{2a}$$

$$\Psi_2(z) = -\frac{1}{2\pi}\int_c^d \frac{\bar{g}(x)}{x-z}\,dx + \frac{1}{2\pi}\int_c^d \frac{x}{(x-z)^2}\,g(x)\,dx - \frac{B}{2\pi}\int_c^d \frac{\bar{g}(x)}{x+z}\,dx - \frac{A}{2\pi}\int_c^d \frac{x}{(x+z)^2}\,g(x)\,dx - \frac{A}{\pi}\int_c^d \frac{x}{(x+z)^2}\,\bar{g}(x)\,dx + \frac{2A}{\pi}\int_c^d \frac{x^2}{(x+z)^3}\,\bar{g}(x)\,dx \quad \text{Re } z > 0$$

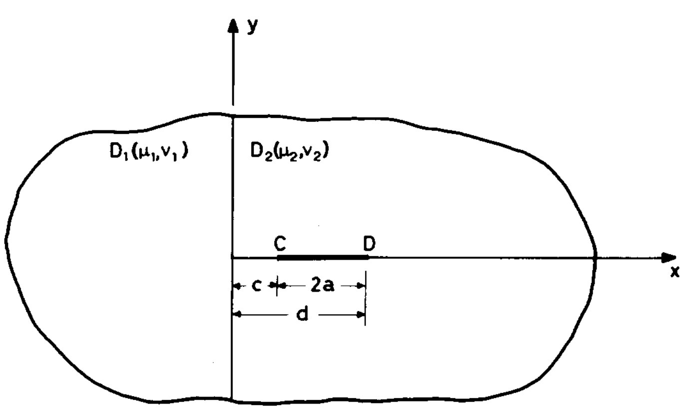


Fig. 1. A crack in a half-plane bonded to a second half-plane consisting of a different isotropic elastic material.

where

$$A = \frac{\mu_2 - \mu_1}{\mu_2 + \kappa_2\mu_1}, \qquad B = \frac{\kappa_1\mu_2 - \kappa_2\mu_1}{\mu_1 + \kappa_1\mu_2} \tag{3}$$

$(d-c) = 2a$ length of the crack

$$\kappa_i = (3-\nu_i)/(1+\nu_i) \quad \text{or} \quad \kappa_i = 3-4\nu_i \quad (i=1,2) \tag{4}$$

for generalized plane-stress or plane-strain conditions, respectively. Finally, the unknown density $g(x)$ is derived from the solution of the following system of singular integral equations:

$$\frac{1}{\pi}\int_c^d \frac{g(x)}{x-x_0}\,dx + \frac{1}{2\pi}\int_c^d H(x,x_0)g(x)\,dx$$

$$= \sigma_{nn}(x_0) + i\sigma_{nt}(x_0), \quad c < x_0 < d \tag{5}$$

$$\int_c^d g(x)\,dx = 0 \tag{6}$$

where relation (6) expresses the condition of single-valuedness of displacements along the crack CD, $\sigma_{nn}(x_0)$ and $\sigma_{nt}(x_0)$ are the normal and shear components of stresses, respectively, applied at the point x_0 of the crack, and

$$H(x,x_0) = \frac{1}{(m+\kappa_1)(1+m\kappa_2)} \tag{7}$$

$$\Big\{[(1+m\kappa_2)(m+\kappa_1) - m(1+\kappa_2)(1+m\kappa_2) - 3(1-m)(m+\kappa_1)] \times \frac{1}{x_0+x}$$

$$+ 12(1-m)(m+\kappa_1)\frac{x_0}{(x+x_0)^2} - 8(1-m)(m+\kappa_1)\frac{x_0^2}{(x+x_0)^3}\Big\}, \quad m = \frac{\mu_1}{\mu_2}$$

The Cauchy type singular integral equation (5) can only be solved numerically.[10] The integrals are approximated by taking into consideration the Lobatto–Chebyshev rule.[11] In what follows, by taking the appropriate collocation points we obtain from the eqs. (5) and (6) a linear system, with unknowns the values of the density $g(x)$ at the integration points $\{x_i\}_{i=1}^{n}$. Therefore, $\Phi_i(z)$ and $\Psi_i(z)$ $(i=1,2)$ can be written as follows:

$$\Phi_1(z) = \frac{1-B}{2\pi}\sum_{i=1}^{n}\frac{\Lambda_i(z)}{x_i - z}g(x_i), \quad \mathrm{Re}\, z < 0 \tag{8a}$$

$$\Phi_2(z) = \frac{1}{2\pi}\sum_{i=1}^{n}\Lambda_i(z)\left[\frac{1}{x_i - z} + \frac{A}{x_i + z} - \frac{2Ax_i}{(x_i+z)^2}\right]g(x_i), \ \mathrm{Re}\, z > 0 \tag{8b}$$

$$\Psi_1(z) = \frac{1}{2\pi}\sum_{i=1}^{n} \Lambda_i(z)\left\{(1-A)\left[\frac{x_i}{(x_i-z)^2} - \frac{1}{x_i-z}\right] - \frac{(A-B)x_i}{(x_i-z)^2}\right\} g(x_i), \quad \text{Re } z < 0 \tag{9a}$$

$$\Psi_2(z) = \frac{1}{2\pi}\int_{i=1}^{n} \Lambda_i(z)\left[-\frac{1}{x_i-z} + \frac{x_i}{(x_i-z)^2} - \frac{B}{x_i+z} - \frac{3Ax_i}{(x_i+z)^2} + \frac{4A}{(x_i+z)^3}x_i^2\right] g(x_i), \quad \text{Re } z > 0 \tag{9b}$$

where $\Lambda_i(z)$ are the "weights" of the integration rule and

$$x_i = \cos\frac{i-1}{n-1}, \quad i = 1(1)n \tag{10}$$

the "integration points."

FRACTURE CRITERIA IN GLASSY POLYMERS

The Mises criterion, which has been successfully used in metals, encounters a lot of problems in glassy polymers. This occurs because the yield locus of glassy polymers basically depends on the hydrostatic stress component which is not taken into account in Mises criterion, and also because the glassy polymers present different yield locus in tension and compression. So, the yield locus in glassy polymers does not coincide with the yield locus resulting from the criterion of Mises. In order to overcome this difficulty, many efforts have been made ending in two main criteria which take into consideration Mises criterion. The first criterion, applied in glassy polymers by Bauwens[15] and Sternstein and Ongchin,[16] was initially proposed by Nadai[17] and has the general form

$$\frac{1}{2^{1/2}}[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2]^{1/2} + \frac{R-1}{R+1}(\sigma_1+\sigma_2+\sigma_3) = \frac{2R}{R+1}\sigma_y^t \tag{11}$$

where

$$R = \sigma_y^c/\sigma_y^t$$

with σ_1, σ_2, σ_3 the principal stresses and σ_y^c, σ_y^t the absolute values of the compressive and tensile yield strengths, respectively, of the material. In what follows this criterion will be referred as the *first modified Mises criterion* (MM1).

The second criterion was proposed by Schleicher (see Ref. 14) and has been developed by Meldahl.[14] This criterion referred in what follows as the *second modified-Mises criterion* (MM2) is given by the formula

$$(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^3 + (\sigma_3-\sigma_1)^2 + 2(\sigma_y^c-\sigma_y^t)(\sigma_1+\sigma_2+\sigma_3) = 2\sigma_y^c\sigma_y^t \quad (12)$$

It can be observed that both criteria are pressure-dependent and degenerate to the Mises criterion if the yield strengths in tension and compression are equated ($\sigma_y^c = \sigma_y^t$, i.e., $R=1$). It must also be noted that the predictions of the second criterion for the elastic–plastic boundaries are much closer to the experimentals findings than the predictions of the first one.

The distribution of stresses σ_{nn} and σ_{nt} applied to both lips of the crack causes a high stress concentration in a small neighborhood of the crack tips, and as a consequence plastic strains will develop around the crack tips. But, the region in which plastic strains will develop is not known in advance. A first estimation of this region can be made if it is considered that the contour of the plastic region is geometrically similar to that obtained by relation (11) or by relation (12) and that the principal stresses are yielded using the Muskhelishvili relations[18] with the aid of complex potentials $\Phi_i(z)$ and $\Psi_i(z)$ $(i=1,2)$. So, the stresses are only confined in their elastic components, and therefore the resulting elastic-plastic boundaries are valid for the first step of deformation of the plate. Finally, the first criterion (MM1) takes the form

$$\{4[\mathrm{Re}\,\Phi_i(z)]^2\,[1+4\nu(\nu-1)]+3|\,\bar{z}\,\Phi_i'(z)+\Psi_i(z)|^2\}^{1/2} + 4\,\frac{R-1}{R+1}(1+\nu)\,[\mathrm{Re}\,\Phi_i(z)] = \frac{2R}{R+1}\sigma_y^t, \quad i=1,2 \quad (13)$$

and also the second (MM2) criterion gives

$$4[\mathrm{Re}\,\Phi_i(z)]^2[1+4\nu(\nu-1)]+3|\bar{z}\Phi_i'(z)+\Psi_i(z)|^2 + 4(R-1)\sigma_y^t(1+\nu)[\mathrm{Re}\,\Phi_i(z)] = R\cdot\sigma_y^{t2}, \quad i=1,2 \quad (14)$$

where only plane-strain $[\sigma_3 = \nu(\sigma_1+\sigma_2)]$ conditions were considered and $\mathrm{Re}\,\Phi_i(z)$ denotes the real part of $\sigma_i(z)$.

Equations (13) and (14) are solely functions of z [since $\Phi_i(z)$ and $\Psi_i(z)$ are given by relations (8) and (9), respectively]. Let $z^* = x^* + iy^* = r^*e^{i\theta}$, the complex coordinates which satisfy anyone of the above equations. The determination of complex coordinates z^* can be made either by using methods for finding roots or considering the polar angle θ as known and using any Newton method in order to find the only unknown r^*.

The set of complex coordinates z^* determines the position of the incipient elastic plastic boundary between elastic and plastic regions at the crack tip C (the same work may takes place around the crack tip D). But, as our tests have proved when the applied stresses p are less than half of σ_y^t, the elastic–plastic boundary is very small and we can consider that it coincides with the exact one. This occurs because the needed redistribution of stresses after the formation of the first plastic strains is not very important since the glassy polymers are also strongly hardened materials.

RESULTS AND DISCUSSION

The proposed theory was applied to two problems in plane-strain conditions when a constant pressure p ($\sigma_{nn} = p$) is applied to both lips of the crack. The first problem concerned an epoxy–aluminium bimaterial plate (the crack being in the epoxy). The second problem concerned an epoxy–epoxy bimaterial plate (the crack being in the less resisting epoxy). These part of materials were preferred because they correspond to problems usually confronted in practice with composite materials.

In the first problem the Poisson ratios of the two materials were $\nu_1 = 0.30$ and $\nu_2 = 0.35$, respectively, and the ratio of the corresponding shear moduli was $\mu_1/\mu_2 = 23.08$. In the second problem the Poisson ratios of the two materials were $\nu_1 = 0.338$ and $\nu_2 = 0.43$, respectively, and $\mu_1/\mu_2 = = 1.938$. These two epoxy materials correspond to the 100-0-8 and 100-40-8 phr (parts per hundred resin), respectively. An epoxy material with a greater percentage in plasticizer becomes more viscoelastic, and therefore the elastic solution applied with MM1 and MM2 criterion is not valid. Figures 2(a)–2(f) give the plastic zones in the first problem and for the following values of c/a: 15.00, 1.00, 0.50, 0.25, 0.10, 0.04. Figures 3(a)–3(f) give the plastic zones in the second problem and for the same values of c/a as in Figures 2. From the series of Figures 2 and 3 we can see the transformations taking place in plastic zones as the crack approaches perpendicularly the bimaterial interface. The curves of the figures were plotted for the following values of the constant $R = 1.0$, 1.2, and 1.5. In these figures the plastic enclaves of the upper halves correspond to the MM1 criterion, whereas the plastic enclaves of the lower halves correspond to the MM2 criterion. For $R = 1$ the elastic–plastic boundaries correspond to the Mises criterion. In relations (13) and (14) σ_y^t is equal to $3p$. So, q (see Ref. 3) is not essentially different from unity and therefore can be neglected. From the plastic zones given in Figures 2 and 3 the influence of the reinforcing material while the crack is approaching the bimaterial interface, is shown.

As we can see from these figures, the volume of the plastic zone as well as its width along the axis of the crack for $c/a > 0.10$ monotonically decreases as the crack approaches the bimaterial interface. For values $c/a < 0.10$ an increase of the width of the plastic zone is noted which becomes more important as c/a becomes smaller and R increases. It is interesting to note that the minimum of the width is not localized, but it was observed that for $c/a = = 0.04$ greater values for it are obtained (in the case of an epoxy–epoxy bimaterial plate and for $R = 1$ this remark is not valid). On the contrary, the volume of the plastic zone given by the MM1 criterion decreases monotonically. In the case of MM2 criterion, the above remark is true only for $R < 1.5$. That means, that for $R = 1.5$ and $c/a = 0.04$ greater volume of the plastic zone is foreseen than for $c/a = 0.10$ when the MM2 criterion is used. Moreover, for $c \to 0$ the shape of the plastic zone is gradually distorted, becoming more elongate in front of the crack. These phenomena are stronger when the reinforcing material is stiffer (in the case of epoxy–aluminium).

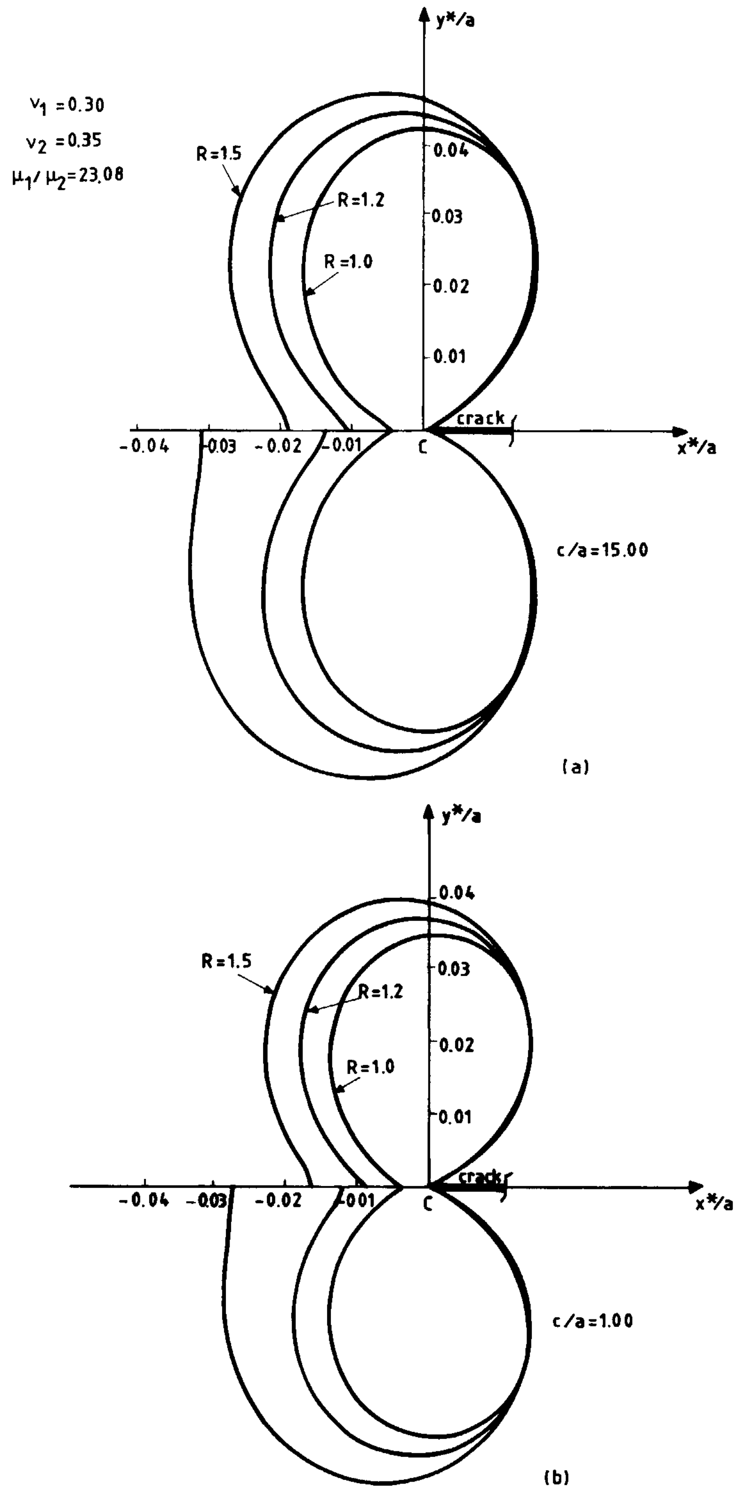
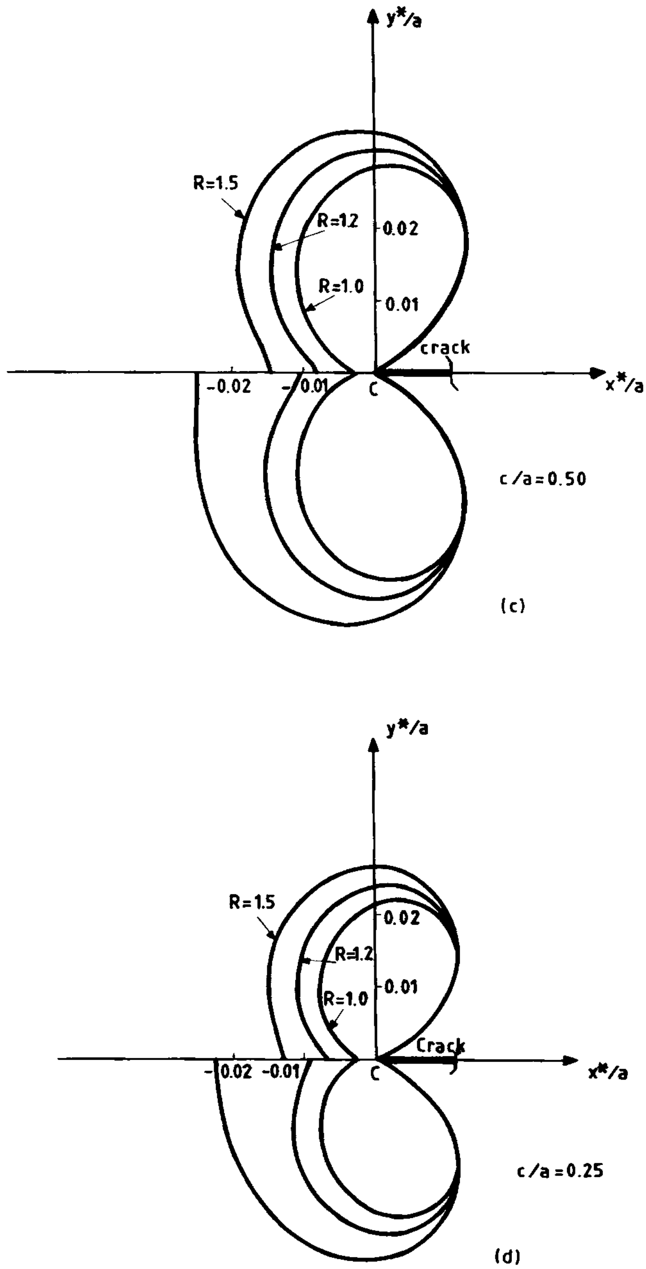


Fig. 2. Elastic–plastic boundaries around the tip C of the crack CD in an epoxy half-plane bonded to an aluminium one. Upper curves correspond to the MM1 criterion while lower ones to the MM2 criterion.



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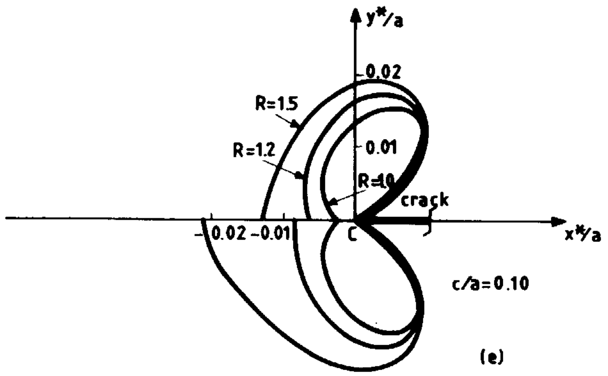

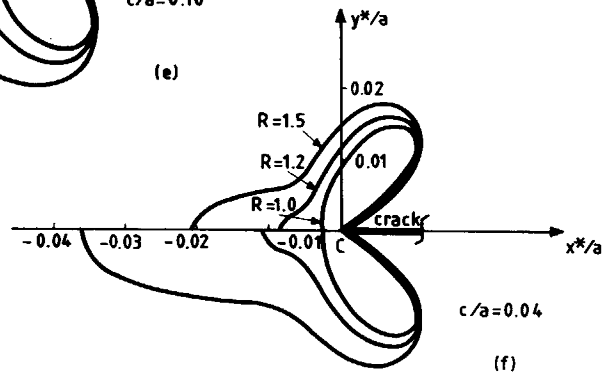

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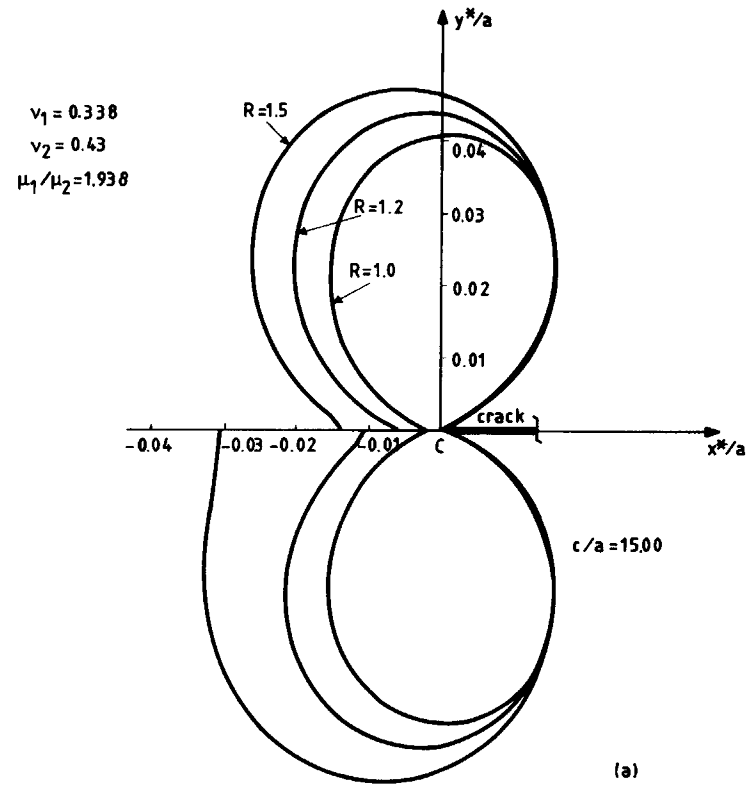

Fig. 3. Elastic–plastic boundaries around the tip C of the crack CD in an epoxy half-plane bonded to a more resisting epoxy one. Upper curves correspond to the MM1 criterion while lower ones to the MM2 criterion.

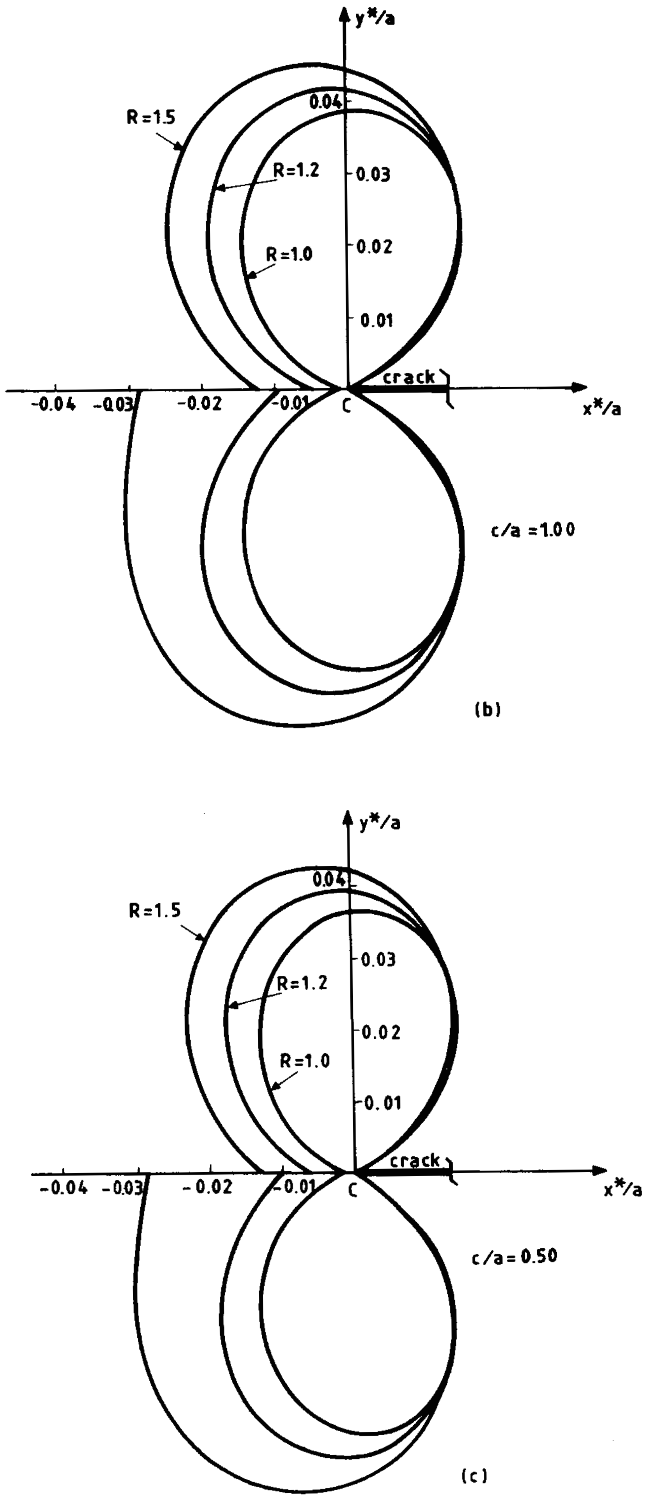

Fig. 3. *(continued from previous page)*

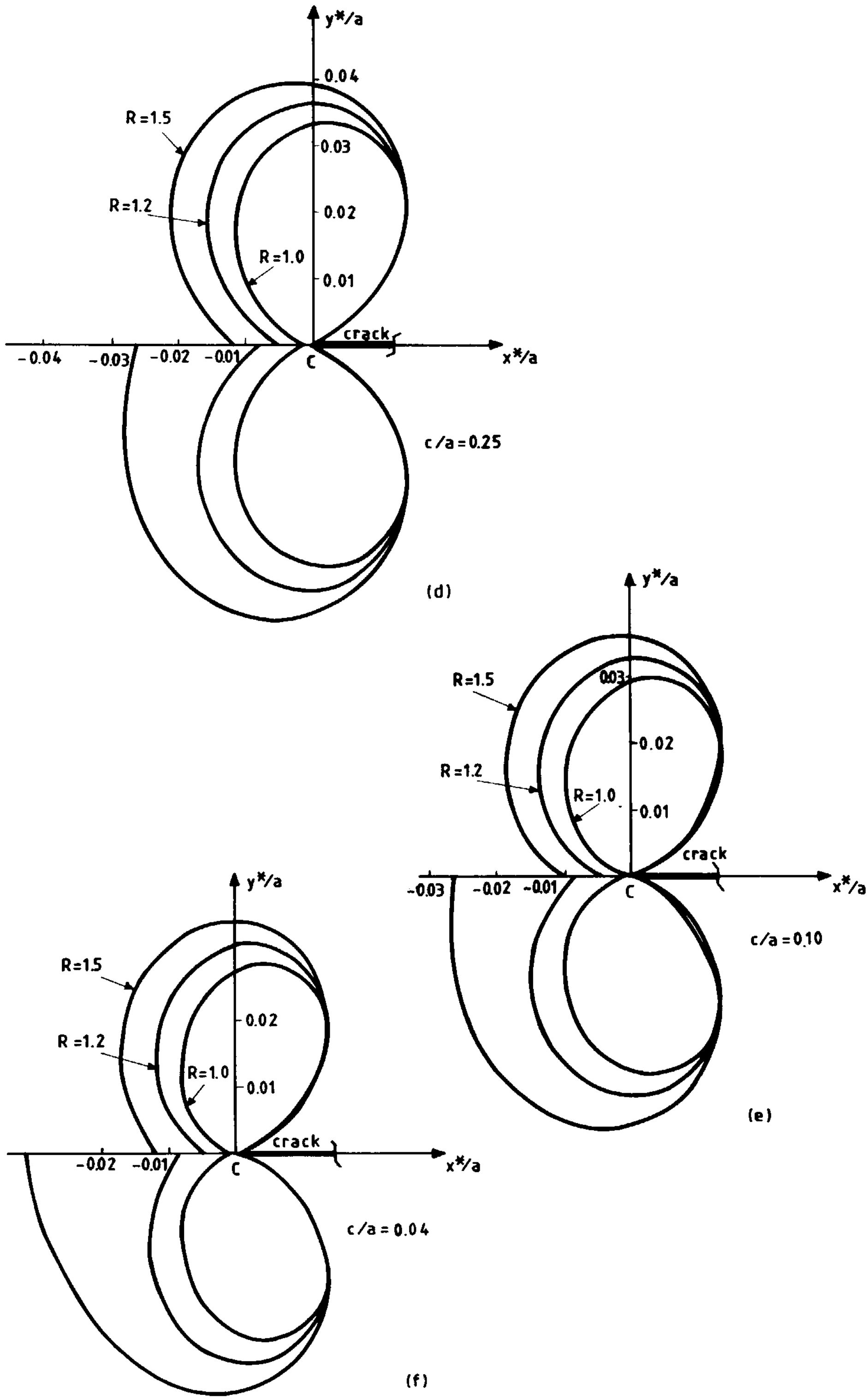

Fig. 3. *(continued from previous page)*

On the contrary, if the crack is in the stiffer material, then, as the crack approaches, the bimaterial interface the plastic zone increases monotonically. Obviously, if the crack is near the boundary, the plastic zone will be expended into the other material.

Another interesting result is that the width of the half of the plastic zone presents two extrema, one of which coincides with the x-axis. The other extremum forms an obtuse angle with the crack for the greater values of c/a which decreases and becomes acute when $c/a \to 0$.

From Figures 2 and 3 it can also be observed that both criteria (MM1 and MM2) present different plastic zones than those presented by the Mises criterion and that these differences increase as R increases.

In the case where $c/a = 15.0$ (the crack being far away to the interface boundary) the plastic zones derived around the crack tip C almost coincides to those of a crack in an infinite epoxy plate. The small differences existing in Figures 2(a) and 3(a) are principally due to the different Poisson ratios of the two epoxy plates. The results derived from Figures 2(a) and 3(a) contradict with the conclusions of Ref. 4 obtained by using Sneddon's asymptotic formula, that "the plastic zones obtained by both criteria are always larger than those obtained by the Mises criterion." On the contrary, it remains true (see Ref. 4) that the plastic zones obtained by the second criterion are always larger than those of the first one. From the Figures 2 and 3 it is also shown that this difference increases as R increases.

Another difference which take place between the exact results and the approximate ones is presented in Figure 4. In it a distortion of the shape

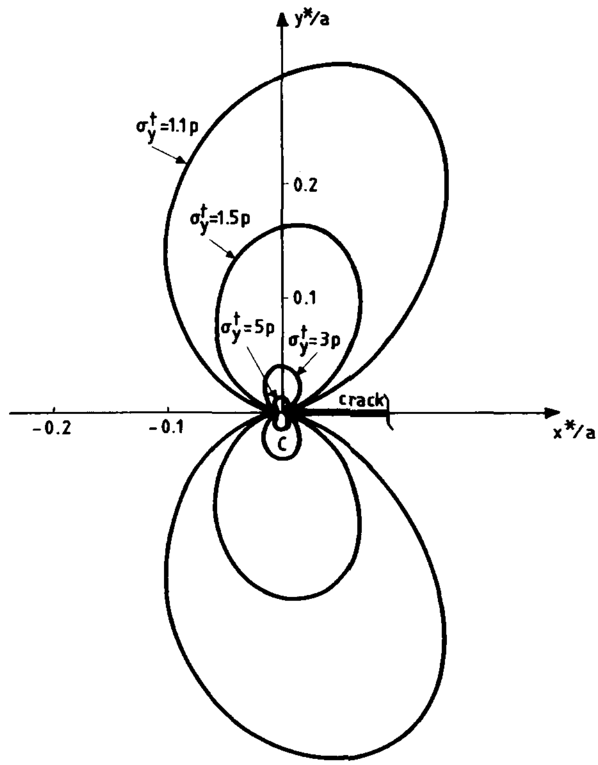

Fig. 4. Mises' elastic–plastic boundaries in an infinite plate for various values of σ_y^t/p.

of plastic zones predicted by the Mises criterion ($R=1$) in plane-strain conditions is observed as the ratio σ_y^t/p decreases. On the other hand, the curves are almost geometrically similar for values of σ_y^t/p greater than 3. Therefore, the plastic zones obtained by the Mises criterion are not always geometrically self-similar for various values of the ratio σ_y^t/p as considered in previous investigations. This can be explained because exact and asymptotic solutions coincide only in a small neighborhood of the crack tip. As the plastic zone becomes greater for values of σ_y^t/p tending to unity, the elastic–plastic boundary obtained by using the asymptotic solution is not reliable.

To see the differences between the exact and the approximate formulas, the plastic zones predicted by the Mises criterion in plane-strain conditions are plotted in Figure 5 in the case of an epoxy plate with $\nu = 0.43$ and $E = 1.82$ GPa correspond to 100-40-8 phr, using first the exact formulas

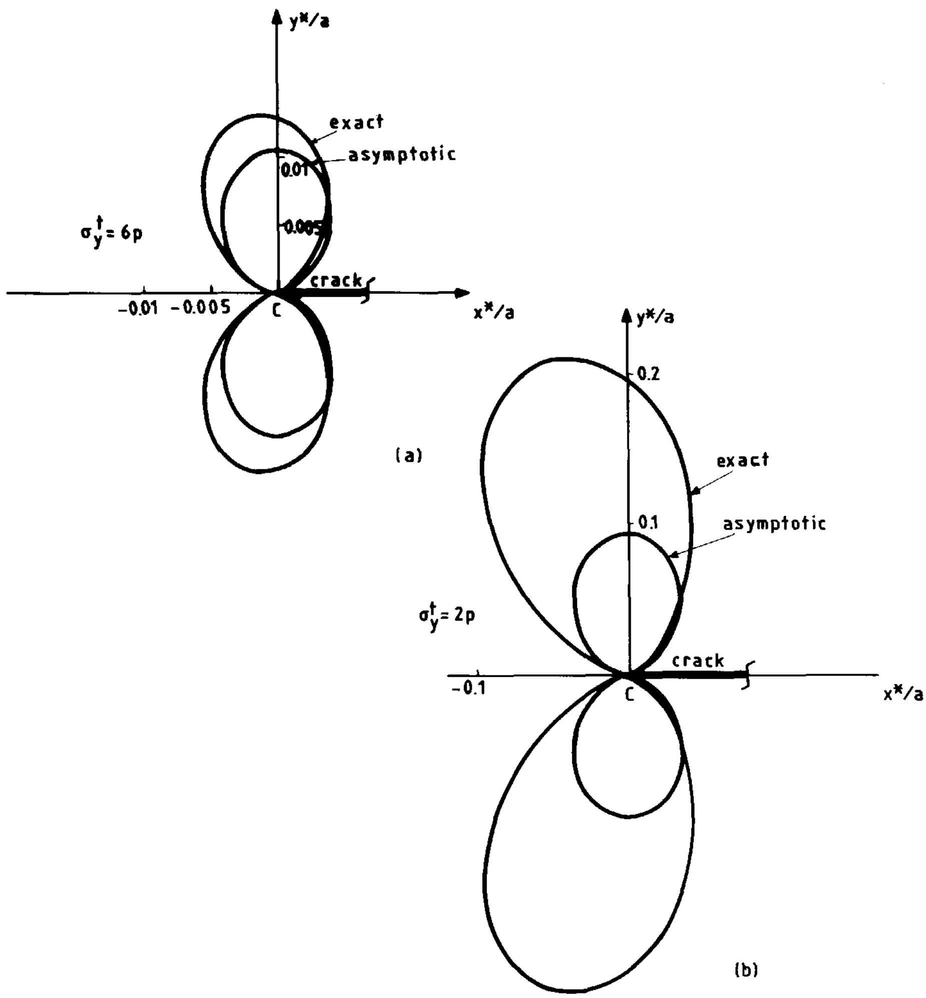

Fig. 5. Mises' elastic–plastic boundaries in an infinite plate subjected to tractions p at infinity, using either the exact solution of the elastic problem or the asymptotic one, when $\sigma_y^t/p = 6.0$ and $\sigma_y^t/p = 2.0$, respectively.

and second the Sneddon asymptotic one for $\sigma_y^t/p=6.0$ and $\sigma_y^t/p = 2.0$, respectively, where p is a uniform traction at infinity, contrary to previous results, which are obtained for p applied on the crack lips. From Figure 5 it is shown that the plastic zones obtained from the asymptotic solution are, for the various values of σ_y^t/p, self-similar whereas the shape of the plastic zones derived from the exact numerical solution presents significant distortions. Moreover, there is a great difference between the exact and the asymptotic plastic zones for $\sigma_y^t/p = 2.0$. Therefore, the use of asymptotic solution for σ_y^t/p tending to unity is unreliable.

It is interesting to note that the above remarks are also valid for the case of plane-stress conditions.[6] Another interesting remark is the similarity of the shape of the plastic zone in both cases (plane-strain conditions or plane-stress ones) when the crack is near the interface. This similarity does not exist when the crack is sent away from the boundary.

CONCLUSIONS

In this paper the plastic zones developed around the tip of a crack under opening-mode loading conditions in glassy polymers when the crack approaches perpendicularly a bimaterial interface were studied. For the prediction of the plastic zones, two pressure-modified Mises yield criteria, MM1 and MM2, respectively, and the exact solution of the problem were used. From the examples considered the following conclusions are derived:

1. The crack-tip plastic zones derived from the MM1 and MM2 criteria are different and not always larger than those predicted by the Mises criterion.
2. The plastic zones predicted by the second modified criterion are always larger than those of the first one. This difference increases as R increases.
3. A distortion of the shape of plastic zones as the crack approaches the bimaterial interface is noted.
4. The plastic zones predicted by the Mises criterion for small values of the ratio σ_y^t/p present significant differences. As the ratio σ_y^t/p increases the plastic zones become self-similar.
5. Great differences in the shape of plastic zones using the exact formulas and the asymptotic ones for small values of the ratio σ_y^t/p are also seen.
6. The plastic zones, when the crack approaches the bimaterial interface, i.e., the stiffer material, becomes smaller and more elongated along the x-axis. This could have been predicted since the reinforcing material hinders the deformation of the other half-plane, and for the creation of plastic zones large deformation is needed. The fact that the plastic zone becomes smaller is an advantage for our analysis because the above plastic zone cannot be deformed essentially when a redistribution of stresses would be considered. So, it can be said that the shape of plastic zones when the crack is near the interface is similar to that given by the elastic–plastic solutions.

References

1. P. S. Theocaris and G. Milios, *Int. J. Fracture,* **16**, 31 (1980).

2. F. A. McClintock and G. Irwin, *Fracture Toughnes and Its Applications,* ASTM STP 381, Am. Soc. for Testing and Mater., Philadelphia, 1965, pp. 84–113.

3. P. M. Vitvitskii, V. V. Panasguk, and S. Ya. Yarema, *Strength Mater. (Problems Procknosti),* **5**, 135 (1971).

4. E. E. Gdoutos, *J. Appl. Polym. Sci.*, **26**, 1919 (1981).
5. E. E. Gdoutos, *J. Appl. Polym. Sci.*, **27**, 879 (1982).
6. E. E. Theotokoglou, G. Tsamasphyros, and P. S. Theocaris, in Proceedings, Congress on Interrelations between Processing Structure and Properties of Polymeric Materials (IUPAC), J. C. Seferis and P. S. Theocaris, Eds., Elsevier, New York, 1984, pp. 423–434.
7. T. S. Cook and F. Erdogan, *Int. J. Eng. Sci.*, **10**, 677 (1972).
8. G. Tsamasphyros and P. S. Theocaris, *Ing. Arch.*, **52**, 159 (1982).
9. N. I. Ioakimidis and P. S. Theocaris, *Int. J. Fracture*, **15**, 299 (1979).
10. P. S. Theocaris and G. Tsamasphyros, *Appl. Anal.* **9**, 37 (1979).
11. D. F. Paget and D. Elliott, *Numer. Math.*, **19**, 373 (1979).
12. R. S. Raghava and R. M. Caddell, *Int. J. Mech. Sci.*, **15**, 967 (1973).
13. P. S. Theocaris, *Int. J. Mech. Sci.*, **18**, 171 (1976).
14. R. S. Raghava, R. M. Caddell, and G. S. Y. Yeh, *J. Mater. Sci.*, **8**, 225 (1973).
15. J. C. Bauwens, *J. Polym. Sci., Part A-2,* **8**, 893 (1970).
16. S. Sternstein and L. Ongchin, *Am. Chem. Soc., Polym. Prepr.*, **10**, 1117 (1969).
17. A. Nadai, *Theory of Flow and Fracture of Solids,* McGraw-Hill, New York, 1950, pp. 210–220.
18. N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity,* Noordhoff, Groningen, Netherlands, 1963, pp. 515–524.

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